Magnetic Surface Breaking by Islands

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Abstract. Analytical M.H.D toroïdal equilibria with flow may be obtained. Magnetic configurations corresponding to small pertubations of these equilibria are numerically studied. A model for transport is proposed.

Introduction. It is well known that the breaking of magnetic surfaces in a toroïdal plasma by resonant magnetic perturbations are related to the growth of magnetic islands. Thus the rather chaotic behaviour of field lines can affect considerably the radial particle transport. We have considered the influence of plasma rotation on the behaviour of excited modes by given small force free magnetic field, analytic solutions of equilibrium with flow being given along lines analogous to (1) (2). The stochasticity threshold, as determined numerically, can also be estimated using results from renormalization techniques in hamiltonian formalism recently developed in (7) (8). Finally, in a rather phenomenological model for stochastic dynamics of particles in such magnetic configurations, we obtain radial diffusion rate with 1/B magnetic field behaviour.

I. Small magnetic perturbations of a toroïdal plasma equilibrium with flow.

In a recent paper (1) (see also (2)), it has been constructed exact solutions of the stationary MHD equations for a rotating axisymmetric plasma. Here, in a first step, we give others analytic solutions in the case of a purely toroïdal flow.

We consider the Shafranov type equation (2)

$$\mathcal{L}(\psi) + J(\psi)J'(\psi) + r^2\left(1 + \frac{r^2\Omega^2}{2r_0^2}\right)^{\frac{\gamma}{\gamma-1}} G(\psi) = 0$$

where, in cylindrical coordinates (r, φ, z), $\mathcal{L} = \frac{\partial^2}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)$, ψ is the poloïdal magnetic flux, $\gamma = 5/3$, $G(\psi)$ is a functional which tends to $p'(\psi)$, the derivative of the pressure functional, as Ω tends to zero. Here, Ω measures the flow, namely $v_\varphi^2/r^2(\gamma e - \frac{v_\varphi^2}{2}) = \Omega^2/r_0^2$, e being the internal energy and r_0 a scale length parameter. $J(\psi)$ gives the toroïdal component of the magnetic field $\vec{B} = (J(\psi)\vec{e}_\varphi + \vec{e}_\varphi \wedge \vec{\nabla}\psi)/r$ (this equation has been derived when the entropy is chosen as a ψ functional).

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We have done the following choice : (3) $G(\psi) = \frac{P}{r_0^4}$ and $J^2(\psi) = J_0^2 + \frac{1}{r_0^2}(\psi^2 - \psi_0^2)$ P, J_0 and ψ_0 being some constants. The solution is (see also (4) for $\Omega = 0$)

$$\psi(r,z) = g_0 \cos\left(\frac{z}{r_0}\right) + r\left(\phi_0 J_1\left(\frac{r}{r_0}\right) + \phi_1 N_1\left(\frac{r}{r_0}\right)\right) + 2P \sum_{n \geq 0} a_n \left(\frac{r}{r_0}\right)^{2n}$$

J_1 and N_1 being Bessel functions of 1st and 2nd kind. All constants depend on Ω, the a_n's being decreasing coefficients. ψ has to verify the following conditions : ψ has a maximum for $z = 0$, $r = r_0$ (which will define the magnetic axis) ; it verifies $\partial^2\psi/\partial r^2 = \xi^2 \partial^2\psi/\partial z^2 \,/\, z = 0, r = r_0$. All coefficients are in terms of g_0, r_0, ξ and Ω. In this way, we have unicity of the magnetic axis within the variable plasma boundary. The pressure being given here by $p = \frac{P}{r_0^4}\left(1 + \frac{\Omega^2 r^2}{2 r_0^2}\right)^{5/2}(\psi - \psi_0)$, $p = 0$ on the boundary (chosen inside the region limited by the separatrix) so $\psi_0 = \psi(r_b, z_b)$; g_0 may be related to the toroïdal component of the current density on the axis by $g_0 = \mu_0 R_0^3 j_0^T/(\xi^2+1)$; J_0 is known in terms of the toroïdal component of the magnetic field B_0. Finally, one has for the field $B_\varphi = \frac{1}{r}\left[B_0^2 r_0^2 - \frac{1}{r_0^2}(g_0^2 - \psi^2)\right]^{1/2}$ $\vec{B}_p = \frac{1}{r}\vec{e}_\varphi \wedge \vec{\nabla}\psi$. The breaking of these magnetic surfaces by perturbations which don't preserve the toroïdal (φ) invariance is well known from a theoretical point of view (5) (6) (7) (see also (8)). Here, we have considered the simplest case of perturbing field of the form $\delta\vec{B} = \varepsilon \frac{i}{2}\left(\exp - i\left(\frac{z}{r_0} + \varphi\right)\right)(\vec{e}_r - i\vec{e}_\varphi) + c.c$ and we have chosen B_0 and g_0 in such a way that this perturbation is resonant for $q = 1, 2$ where q is the safety factor.

Numerical integration of field line equations is performed with a Merson routine, on a IBM 3081 Computer ; $Acc = 10^{-4}$ for the required relative accuracy. Results are shown on fig. 3, 4 for $B_0 = 1T$ and $g_0 = 1.04\,Wb$ where the $m = 1$ mode has been excited ; for $\Omega = 0$, the threshold is $\varepsilon \simeq 8.10^{-2}T$. The analogous case , $B_0 = 1T$ and $g_0 = .55\ Wb$, where the $m = 2$ mode is present, is considered in fig. 5, 6. The hamiltonian formalism can be used to give an analytical estimate of the threshold. As is well known, (7), the magnetic line equations may be viewed as the hamilton equations for $H(p, z, \varphi) = \psi(r, z)$ where φ is "time" and the canonical coordinate associated to z is $p(r, z, \varphi) = \int_0^r B_\varphi\, dr$. In the non perturbed case, H_0 is a φ independant one degree of freedom hamiltonian and is thus integrable. So, if (I, θ) are the action-angle variables where $I = \frac{1}{2\pi}\oint p\, dz$, the integral being taken on a given $\psi(r,z) = cte$ curve, one has $H_0 = H_0(I)$, and $d\theta/d\varphi = 1/q(I)$ where q is the safety factor. In our case, $q = \frac{r_0}{g_0}\left(B_0^2 r_0^2 + \frac{2g_0^2}{r_0^2}\left(1 - \frac{\psi}{g_0}\right)\right)^{1/2}$ near the magnetic axis.

It can be shown that the perturbed potential may be well approximated, in a Fourier expansion with respect to these variables, by only two terms. In such a way, after canonical transformations, the hamiltonian H takes the form $H(v, u, \varphi) = \frac{v^2}{2} - \alpha \cos u - \beta \cos(2u + \varphi)$ where $\alpha = 4\varepsilon\, |H_0''(I_1) V_1|$, $\beta = 4\varepsilon\, |H_0''(I_1) V_2|$ and V_1, V_2 are Fourier coefficients evaluated for $I = I_1$ (I_1 toroïdal flux accross the $q = 1$ surface). It has been considered in (7) (8) the growing of the stochastic layer which appears around the separatrix corresponding to the hamiltonian with $\beta = 0$. The obtained threshold is $\xi_3 = .017/|\epsilon_1 \psi_1|$ which for $\Omega = 0$, $m = 1$ gives $\xi_3 = .12\tau$.

II. A model for stochastic dynamics.

The dynamics of the particles in magnetic structure consisting of numerous small islands (many excited m and n modes) may be viewed, in the collisional case, as the dynamics of stochastic, non isotropic diffusion processes, namely verifying some stochastic differential equation of the type

$$dX_t = \beta(X_t, t)\, dt + D\sigma\, dW_t$$

W_t is the standard Brownian motion, $D^2 = \frac{kT\tau}{\mu}$, μ is the electron mass, τ is a characteristic time of the diffusion inversely proportional to the viscosity. In the slab geometry we consider here, with the magnetic field essentially over the z axis, σ is a symmetric constant matrix $\sigma = \begin{bmatrix} \sigma_x & \sigma^* & 0 \\ \sigma^* & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$; β is a drift term which has to be determined by some stochastic dynamical assumption (9) (10) (11) (12). One defines the derivative of functional of the process as $D_\pm F(x,t) = \lim_{\Delta t \to 0} \pm(\Delta t)^{-1} E\left(F(X_{t\pm\Delta t}, t \pm \Delta t) - F(X_t, t) \,/\, X_t = x\right)$

thus $D_\pm F = \partial_t F + \beta_\pm^i \partial_i F \pm \frac{D^2}{2} g^{ij} \partial_i \partial_j F$ where $g = \sigma\sigma^T$, $\beta \equiv \beta_+$

and $\beta_-^i = \beta_+^i - D^2 g^{ij} \partial_j \operatorname{Log} \rho$. Here ρ is the distribution probability of the process. One has $D_\pm X(x,t) = \beta_\pm(x,t)$, so: β_+ and β_- (in general different), have the meaning of mean velocities. It can be shown that $v = \frac{1}{2}(\beta_+ + \beta_-)$ verifies the equation of continuity $\partial_t \rho + \nabla.(\rho v) = 0$ while $u = \frac{1}{2}(\beta_+ - \beta_-)$ is given by $u^i = \frac{D^2}{2} g^{ij} \partial_j \operatorname{Log} \rho$.

Now, the stochastic dynamical assumption (9) may be given by (for magnetic forces):

$a_{cc} = \frac{1}{2}(D_+\beta_- + D_-\beta_+) = \frac{e}{\mu}\, v \wedge B$. This leads to non linear partial differential equations for u and v, which may be related to simpler one of the type: $\partial_t K(x,y) = \mathcal{D} K(x,y)$ where $\mathcal{D} = \frac{1}{2}(\partial_\ell - \frac{ie}{\mu} A_\ell) G^{\ell m} (\partial_m - \frac{ie}{\mu} A_m)$ where A is the vector potential and G is the (x,y) part of $g = \sigma\sigma^T$. The knowledge of K gives ρ and v, which in turn gives β. In the most simple case (stationary process) one is led to an Ornstein Uhlenbeck process of the type

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix} dt + B \begin{pmatrix} dW_t^x \\ dW_t^y \end{pmatrix}$$

where $A = \frac{|e| B_0}{2\mu} \begin{bmatrix} -\lambda_1/\lambda_2 & -1 \\ 1 & -\lambda_2/\lambda_1 \end{bmatrix}$ and $B = D \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

λ_1^2 and λ_2^2 being eigenvalues of g. Finally, one has

$$\langle X_\tau^2 \rangle / \tau = F(\lambda_1, \lambda_2) \frac{kT}{|e| B} + O(1/\tau)$$

This behaviour corresponds to the simplest solution of the $\mathcal{K}$ equation leading to linear stochastic differential equation wich can be integrated. For others solutions of the $\mathcal{K}$ equation, which can be given, the situation is much more complex, leading in general, to non linear stochastic equations.

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FIGURE CAPTIONS

Fig. 1 Cross section of magnetic surfaces ($\varepsilon = 0$) for $\mathcal{M}_0 = 0$ (full lines) and $\mathcal{M}_0 = 1$ (dotted lines) (Ω is related to the Mach number $\mathcal{M}_0$ for $r = r_0$ by $\Omega^2 = 2\mathcal{M}_0^2 / \left(\frac{2}{\gamma - 1} - \mathcal{M}_0^2\right)$)

Fig. 2 Cross section of pressure surfaces ($\varepsilon = 0$) for $\mathcal{M}_0 = 0$ (full lines) and $\mathcal{M}_0 = 1$ (dotted lines)

Fig. 3 Cross section of perturbed field lines ($\mathcal{M}_0 = 0$, $B_0 = 1T$, $g_0 = 1.04\ Wb$)

a: $\varepsilon = 10^{-3} T$ b: $\varepsilon = 10^{-2} T$ c: $\varepsilon = 4.10^{-2} T$

Fig. 4 Main islands ($m = 1$, $\varepsilon = 10^{-3} T$) for $\mathcal{M}_0 = 0$ (•) and $\mathcal{M}_0 = 1$ (••)

Fig. 5 Cross section of perturbed field lines ($\mathcal{M}_0 = 0$, $B_0 = 1T$, $g_0 = .55\ Wb$)

a: $\varepsilon = 10^{-3} T$ b: $\varepsilon = 10^{-2} T$ c: $\varepsilon = 2.10^{-2} T$

Fig. 6 Main islands ($m = 2$, $\varepsilon = 10^{-3} T$) for $\mathcal{M}_0 = 0$ (•) and $\mathcal{M}_0 = 1$ (••)

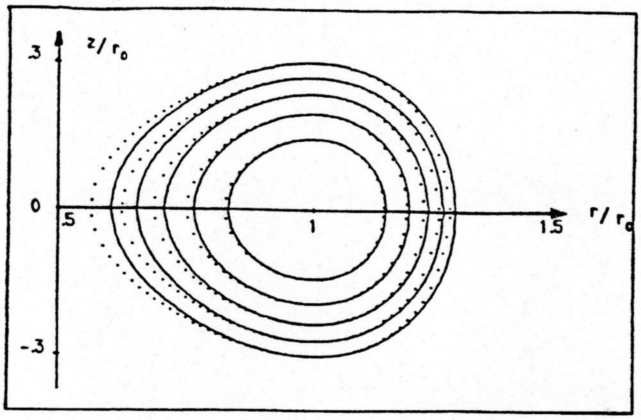
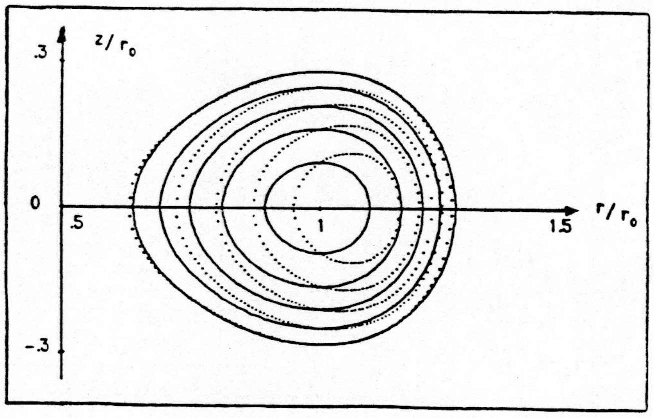


Fig. 1

Fig. 2



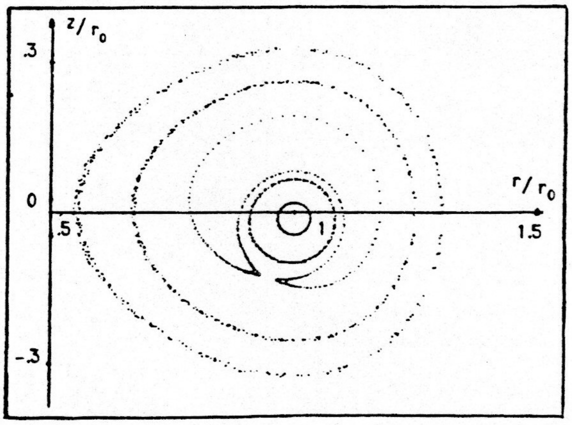


Fig. 3a

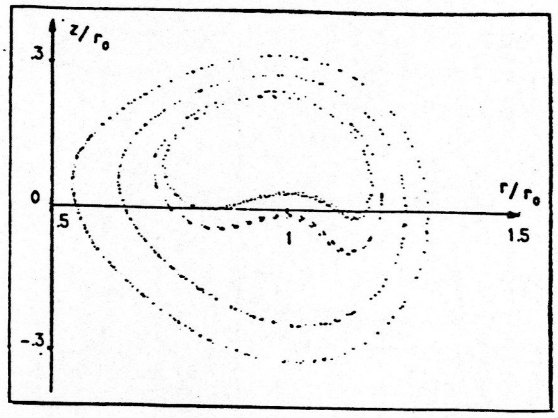


Fig. 3b

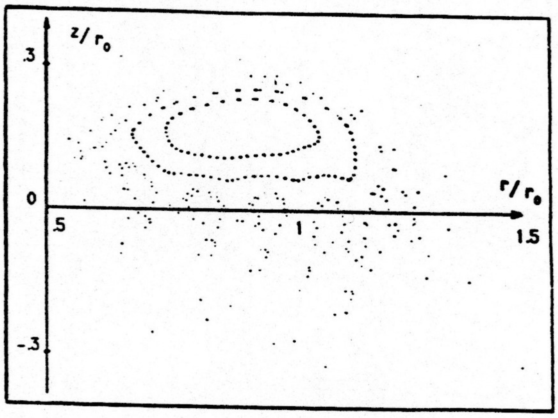


Fig. 3c

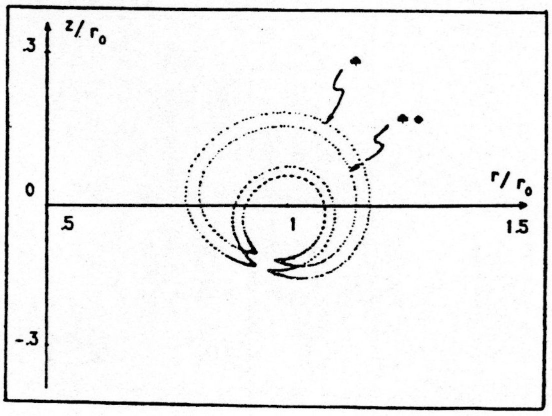


Fig. 4

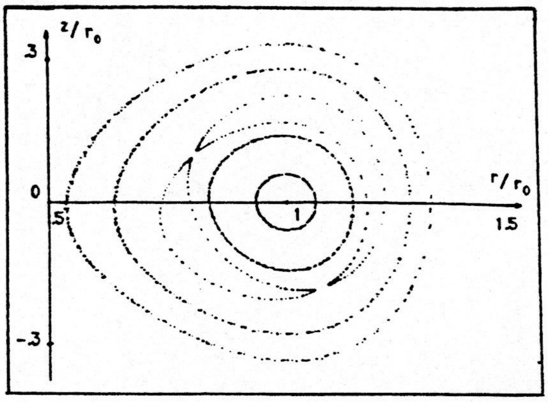


Fig. 5a

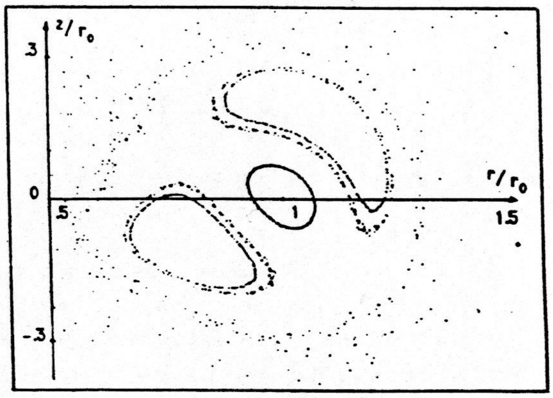


Fig. 5b

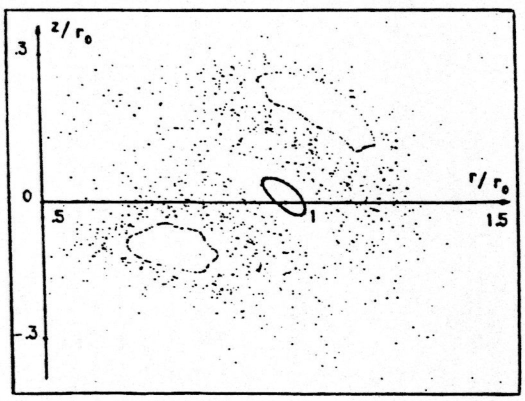


Fig. 5c

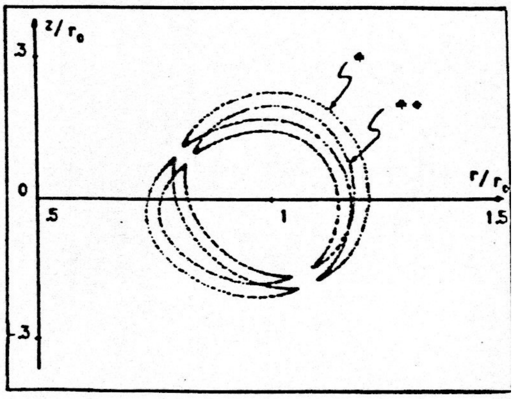


Fig. 6